# Calculus:
# Comparison of the the Disk/Washer and the Shell Methods
## Sandra Peterson, Learning Lab

**Prerequisite Material:** It is assumed that the reader is familiar with the following:

| Method | Axis of Revolution | Formula | Notes about the Representative Rectangle |
|---|---|---|---|
| Disk Method | x-axis | $V = \int_a^b [f(x)]^2 dx$ | $f(x)$ is the length<br>$dx$ is the width |
| | y-axis | $V = \int_c^d [g(y)]^2 dy$ | $g(y)$ is the length<br>$dy$ is the width |
| Washer Method | x-axis | $V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$ | $f(x)$ is the top curve<br>$g(x)$ is the bottom curve<br>$dx$ is the width |
| | y-axis | $V = \pi \int_c^d \left([f(y)]^2 - [g(y)]^2\right) dy$ | $f(y)$ is the right curve<br>$g(y)$ is the left curve<br>$dy$ is the width |
| Shell Method | x-axis | $V = 2\pi \int_c^d y[g(y)] dy$ | $y$ is the distance to the axis of revolution<br>$g(y)$ is the length<br>$dy$ is the width |
| | y-axis | $V = 2\pi \int_a^b x[f(x)] dx$ | $x$ is the distance to the axis of revolution<br>$f(x)$ is the length<br>$dx$ is the width |

It is important to note that the representative rectangle in the Disk and the Washer Methods are always going to be *perpendicular* to the axis of revolution. With the Shell Method, the representative rectangle will always be *parallel* to the axis of revolution.

Another thing that might help while trying to visualize this type of volume problem, is that the Disk Method is used when the representative rectangle produces a solid that is similar to a plate (no hole in the middle). The Washer Method is used when the rectangle sweeps out a solid that is similar to a CD (hole in the middle). And finally, the Shell Method is used when the rectangle sweeps out a solid that is similar to a toilet paper tube.

Think about this: Can you see that the Disk Method is just a specific case of the Washer Method?

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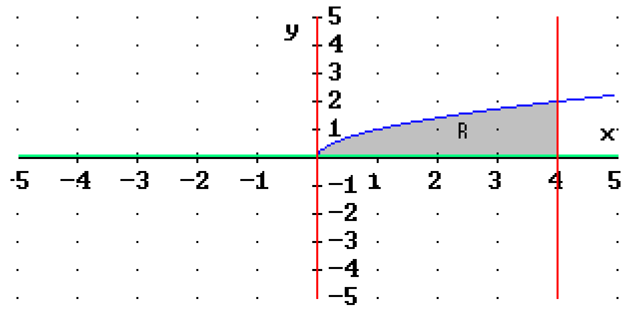
To compare the methods, we will consider the following problem.

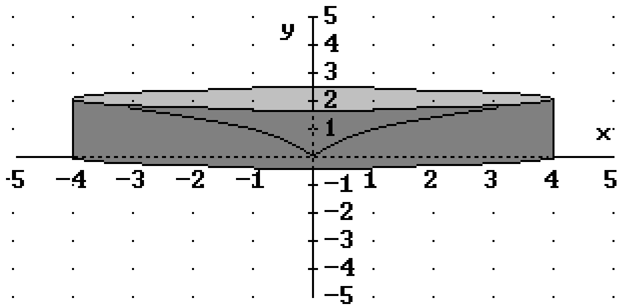
**Problem:** *Sketch the region bounded by the graphs of the given functions for the specified values of x or y. Then find the volume of the solid obtained by revolving the region about the y-axis.*

$y = \sqrt{x}$ $\qquad y = 0$ $\qquad 0 \le x \le 4$

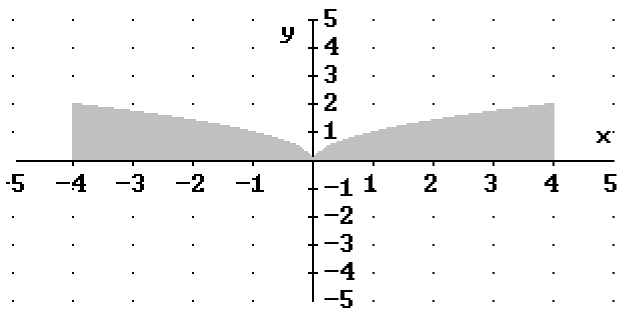
**Preliminary Work:** We begin by considering the graphs of the three given functions: $y = \sqrt{x}$ (shown in blue), $y = 0$ (shown in green), and $0 \le x \le 4$ (shown in red). We'll call the region bounded by these graphs, R (shown in gray).




When we revolve R around the y-axis, we obtain a circular solid. A three-dimensional graph of this might look like the following:




A cross-section of the solid is shown below.

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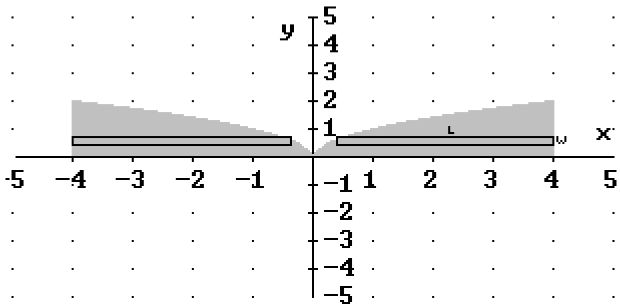
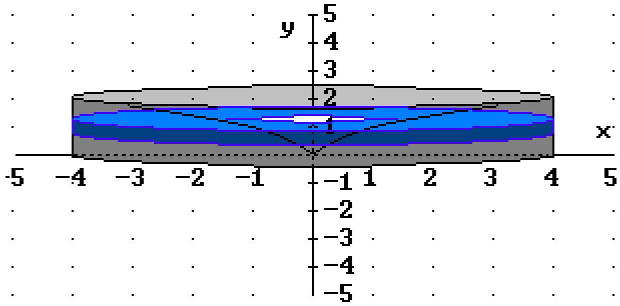
**The Disk/Washer Method:**

The Disk/Washer Method uses representative rectangles that are *perpendicular* to the axis of revolution. Therefore, we have the following:




Or in three-dimensions:




Our formula states: $V = \pi \int_c^d \left( [f(y)]^2 - [g(y)]^2 \right) dy$

where $f(y)$ is the right curve, $g(y)$ is the left curve, and $dy$ is the width.

Using the given information from our problem, we see that our right curve will be $f(y) = 4$ since our right boundary will always be the vertical line $x = 4$. Our left curve is determined by $y = \sqrt{x}$. Since this needs to be a function of $y$, we square both sides of the equations to solve for $x$ and get $x = y^2$. Also, to find the interval of integration, we use the smallest value that $y$ assumes ($c = 0$) and the largest value that $y$ assumes ($d = 2$). Substituting all of these values into our formula, we get:

$$\begin{aligned} V &= \pi \int_c^d \left( [f(y)]^2 - [g(y)]^2 \right) dy \\ &= \pi \int_0^2 \left( [4]^2 - \left[y^2\right]^2 \right) dy \\ &= \pi \int_0^2 \left(16 - y^4\right) dy \\ &= \pi \left[16y - \tfrac{1}{5} y^5\right]_{y=0}^{y=2} \\ &= \pi \left[16(2) - \tfrac{1}{5}(2)^5\right] - \pi\left[16(0) - \tfrac{1}{5}(0)^5\right] \\ &= \pi\left[32 - \tfrac{32}{5}\right] - \pi[0] \\ &= \tfrac{128\pi}{5} \end{aligned}$$

Therefore, the volume of the solid of revolution is $\dfrac{128\pi}{5}$. ■

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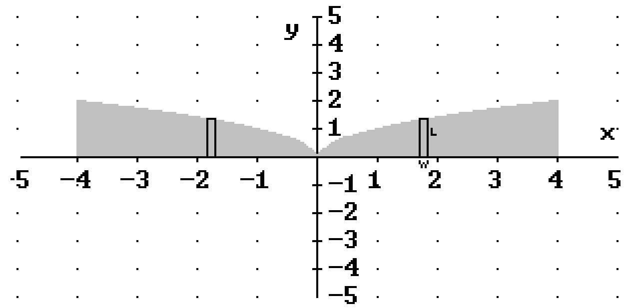
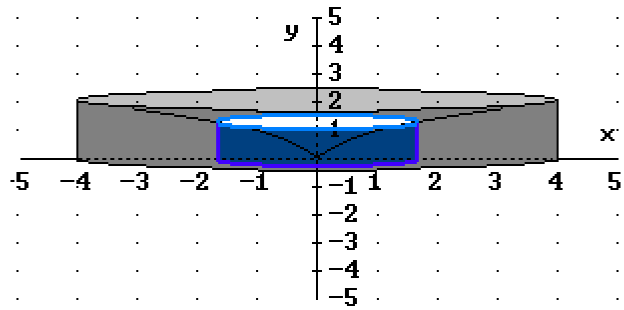
**The Shell Method:**

The shell Method uses representative rectangles that are *parallel* to the axis of revolution. Therefore, we have the following:




Or in three-dimensions:




Our formula states: $V = 2\pi \int_a^b x[f(x)]dx$

where $x$ is the distance to the axis of revolution, $f(x)$ is the length, and $dx$ is the width. Using the given information from our problem, we see that the length of each representative rectangle can be given by $f(x) = \sqrt{x}$. Also, to find the interval of integration, we use the smallest value that $x$ assumes ($a = 0$) and the largest value that $x$ assumes ($b = 4$). Substituting all of these values into our formula, we get:

$$\begin{aligned}
V &= 2\pi \int_a^b x[f(x)]dx \\
&= 2\pi \int_0^4 x\left[\sqrt{x}\right]dx \\
&= 2\pi \int_0^4 x^{\frac{3}{2}} dx \\
&= 2\pi \left[\left(\frac{2}{5}\right)x^{\frac{5}{2}}\right]_{x=0}^{x=4} \\
&= \tfrac{4}{5}\pi\left[(4)^{\frac{5}{2}} - (0)^{\frac{5}{2}}\right] \\
&= \tfrac{4}{5}\pi[32 - 0] \\
&= \tfrac{128\pi}{5}
\end{aligned}$$

Therefore, the volume of the solid of revolution is $\frac{128\pi}{5}$. ■

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